1 Quiz 7 (Mar 25) solutions

• 1. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$
.

First, we need to find the eigenvalues of the matrix, call it A. To do so, we calculate $\det(A - \lambda I) = \det(\begin{bmatrix} 1-\lambda & 2 \\ 0 & -1-\lambda \end{bmatrix} = (1-\lambda)(-1-\lambda) - (0)(2) = (1-\lambda)(-1-\lambda)$ (since $\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$). The zeros of this polynomial (solutions to $(1-\lambda)(-1-\lambda) = 0$) are the eigenvalues, so $\lambda_1 = 1$ and $\lambda_2 = -1$ are our two eigenvalues.

Now, we need to find our eigenvectors (for each eigenvalue λ_i), so we need to solve the linear systems

$$(A - \lambda_i I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

.

First, we handle the case of $\lambda_1 = 1$: note $(A - \lambda_1 I) = \begin{bmatrix} 1-1 & 2 \\ 0 & -1-1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix}$. Now

$$(A - \lambda_1 I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so we have this system of equations to solve for $\begin{bmatrix} x \\ y \end{bmatrix}$:

$$0x + 2y = 0$$

$$0x + (-2)y = 0$$

This system tells us that $y = 0$ (since $2y = 0$), so our λ_1-eigenvectors are $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ for any <u>NONZERO</u> real number x (eigenvectors need to be nonzero). We'll chose $x = 1$ for simplicity, so our λ_1-eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Now, we handle the case of $\lambda_2 = -1$: note $(A - \lambda_2 I) = \begin{bmatrix} 1-(-1) & 2 \\ 0 & -1-(-1) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$. Now

$$(A - \lambda_2 I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so we have this system of equations to solve for $\begin{bmatrix} x \\ y \end{bmatrix}$:

$$2x + 2y = 0$$

$$0x + 0y = 0$$

This system tells us that $2x = -2y$ (or $y = -x$ dividing by 2), so our λ_2-eigenvectors are $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix}$ for any <u>NONZERO</u> real number x (why?). We'll chose $x = 1$ again for simplicity, so our λ_2-eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and we're done: our (eigenvalue,eigenvector) pairs are (1, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$) and (-1, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$).

• 2. Can this system of equations below be represented by a matrix equation? Why or why not?

$$x_1(t+1) = 0.4x_1(1 - \frac{x_1(t)}{100}) - 0.1x_1(t)x_2(t)$$

$$x_2(t+1) = 0.1x_1(t)x_2(t)$$

.
This system cannot be represented by a matrix equation, since it isn't a linear system of equations.

• 3. Find the values x and y so that the matrix equality holds:

$$\begin{bmatrix} 5 & -1 \\ 2x & y \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 9 & 4-y \end{bmatrix}$$

If two matrices are equal, then each corresponding entry of the two are equal, and vice versa. Thus, we need $5 = 5$, $-1 = -1$, $2x = 9$, and $y = 4 - y$. The first two equalities are automatic, and we know that $x = \frac{9}{2}$ if $2x = 9$. To solve $y = 4 - y$ add y to both sides and then divide by 2 to find $y = 2$.

2 Worksheet (Mar 25) solutions

• 1. Find the normalized eigenvector that would describe the equilibrium structure of the system using the matrix $\begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$.

The system is in equilibrium when

$$\begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

In other words, we have found a 1-eigenvector. To find the normalized eigenvector, we use the system of equations

$$0.7x + 0.4y = x$$

$$0.3x + 0.6y = y$$

. Simplifying, we reduce each one (mimicking the way we did it in the quiz above)

$$0.7x + 0.4y = x \to (0.7-1)x + 0.4y = 0 = (-0.3)x + 0.4y$$

,

$$0.3x + 0.6y = y \to 0.3x + (0.6-1)y = 0 = 0.3x + (-0.4)y = 0$$

which are really the same equation (multiplied by -1) (notice in the quiz we also only used one equation, we'll always only need one when λ is an eigenvalue, for us $\lambda = 1$ is an eigenvalue since the matrix is Markov: $\det(\begin{bmatrix} 0.7-\lambda & 0.4 \\ 0.3 & 0.6-\lambda \end{bmatrix}) = (0.7-\lambda)(0.6-\lambda) - (0.3)(0.4) = \lambda^2 - 1.3\lambda + 0.42 - 0.12 = \lambda^2 - 1.3\lambda + 0.3 = (\lambda - 1)(\lambda - 0.3)$ so $\lambda_1 = 1$, $\lambda_2 = 0.3$).

Since $0.3x + (-0.4)y = 0$, get one variable in terms of the other: $0.3x = 0.4y$, so $x = \frac{0.4}{0.3}y = \frac{4}{3}y$. Then, an eigenvector satisfies $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{3}y \\ y \end{bmatrix}$. Choose $y = 3$ for simplicity to obtain the eigenvector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ (this makes normalizing easier).

To normalize $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, find the sum of the components $(4 + 3 = 7)$ and divide by it: our normalized eigenvector is $\begin{bmatrix} \frac{4}{7} \\ \frac{3}{7} \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

• 2. Write a matrix of size 3x3 that is a transfer matrix with two of its entries being 0.

A transfer matrix is a Markov matrix, so our matrix just needs nonnegative entries with columns that sum to 1:

$$\begin{bmatrix} 0.1 & 0.4 & 0 \\ 0.3 & 0.5 & 0 \\ 0.6 & 0.1 & 1 \end{bmatrix}$$

is an example with exactly two entries being 0.

• 3. Suppose that an eigenvector of a transfer matrix is $\begin{bmatrix} 20 \\ 10 \end{bmatrix}$. Write the normalized eigenvector for this matrix. If the population was initially 60, what are the proportions of the population at equilibrium?

Again, sum the entries and divide: $20 + 10 = 30$, so the normalized eigenvector is

$$\frac{1}{30}\begin{bmatrix} 20 \\ 10 \end{bmatrix} = \begin{bmatrix} \frac{20}{30} \\ \frac{10}{30} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

. To find the proportions of the population at equilibrium, multiply 60 by the normalized eigenvector (it contains the proportion data): $60 * \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 40 \\ 20 \end{bmatrix}$, so 40 members of the population are in the first category, while 20 are in the second.

3 Quiz 8 (Apr 1) solutions

• 1. For the matrix $A = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$, find the dominant eigenvalue.

To find the eigenvalues of A, we solve $\det(A - \lambda I) = 0$ for λ: $\det(\begin{bmatrix} 3-\lambda & 1 \\ 4 & 3-\lambda \end{bmatrix}) = (3-\lambda)(3-\lambda) - (4)(1) = \lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1) = 0$. We find that $\lambda_1 = 5$ and $\lambda_2 = 1$ are our two eigenvalues, with $\lambda_1 = 5$ having the largest absolute value. Thus λ_1 is the dominant eigenvalue.

• 2. Cattle on a ranch are divided into calves, yearlings, and adults. Data indicate that 70% of the calves survive the first year to become yearlings, while 80% of yearlings mature into adults. In addition, 90% of adults survive a given year, and an adult female produces a single calf each year. Construct the Leslie matrix, assuming that we census the population after reproduction and count only females.

Let C(t), Y(t), and A(t) be the number of calves at year t. Then (using our two possibilities of yearlings surviving and staying yearlings or yearlings dying), our system of equations should be

$$C(t+1) = A(t)$$

,

$$Y(t+1) = 0.7C(t) + 0.2Y(t)$$

,

$$A(t+1) = 0.8Y(t) + 0.9A(t)$$

if the 20% of yearlings not maturing survive as yearlings (otherwise $Y(t+1) = 0.7C(t)$). Notice that we assume that each adult that survived year t gave birth to a calf during year t+1, since we census the population after reproduction and only count females.

Then, our two possibilities for Leslie matrices are

$$\begin{array}{c} \\ C(t+1) \\ Y(t+1) \\ A(t+1) \end{array} \begin{array}{c} \begin{array}{ccc} C(t) & Y(t) & A(t) \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 0.7 & 0.2 & 0 \\ 0 & 0.8 & 0.9 \end{pmatrix} \end{array}$$

and

$$\begin{array}{c} \\ C(t+1) \\ Y(t+1) \\ A(t+1) \end{array} \begin{array}{c} \begin{array}{ccc} C(t) & Y(t) & A(t) \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 0.7 & 0 & 0 \\ 0 & 0.8 & 0.9 \end{pmatrix} \end{array}$$

- 3. Is $\begin{bmatrix} 0.5 & 0.5 \\ 1.2 & -0.2 \end{bmatrix}$ a Markov matrix?

No, while the columns sum to 1, a Markov matrix has all nonnegative entries.

4 Worksheet (Apr 1) solutions

• 1. A population fits the Leslie matrix model with four life stages. The dominant eigenvalue is computed to be 1.2, and a corresponding eigenvector is

$$\begin{bmatrix} 3 \\ 7 \\ 10 \\ 2 \end{bmatrix}$$

. If at timestep 500 the population has reached the long-term growth rate with 6400 individuals, how many individuals will be in each class at time step 501?

Normalize the eigenvector to obtain the long-term population structure: again, add the components and divide by the sum $3 + 7 + 10 + 2 = 22 \to \frac{1}{22} \begin{bmatrix} 3 \\ 7 \\ 10 \\ 2 \end{bmatrix}$. The number of individuals in each class at time step 500 is thus $6400 * \frac{1}{22} \begin{bmatrix} 3 \\ 7 \\ 10 \\ 2 \end{bmatrix} \approx \begin{bmatrix} 872 \\ 2036 \\ 2909 \\ 581 \end{bmatrix}$ since the population at that step is 6400. At time step 501, the population has grown by 20% (since the dominant eigenvalue is 1.2 and we've reached the long-term growth rate), so our new population is $1.2 * 6400 = 7680$. The number of individuals in each class at time step 501 is thus

$$7680 * \frac{1}{22} \begin{bmatrix} 3 \\ 7 \\ 10 \\ 2 \end{bmatrix} \approx \begin{bmatrix} 1047 \\ 2443 \\ 3490 \\ 698 \end{bmatrix}$$

- 2. For the matrix $A = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$ (from the quiz) find a formula for $A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

We use the formula for $x(k) = A^k x(0)$ given by $x(k) = c_1(\lambda_1)^k v_1 + c_2(\lambda_2)^k v_2$ where (λ_1, v_1), (λ_2, v_2) are the (eigenvalue, eigenvector) pairs of A. Note that from the quiz, we know our eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 1$.

We now find eigenvectors for these: for $\lambda_1 = 5$, we solve the system $(A - 5I) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for a nonzero $\begin{bmatrix} x \\ y \end{bmatrix}$.

Since $A - 5I = \begin{bmatrix} 3-5 & 1 \\ 4 & 3-5 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$, from $(A - 5I) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we obtain the equations

$$-2x + y = 0$$

$$4x - 2y = 0$$

. Thus $y = 2x$, so a λ_1-eigenvector satisfies

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix}$$

. Choosing $x = 1$, we have found the (eigenvalue, eigenvector) pair $(5, \begin{bmatrix} 1 \\ 2 \end{bmatrix})$.

For $\lambda_2 = 1$, note $A - 1I = \begin{bmatrix} 3-1 & 1 \\ 4 & 3-1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, so from $(A - 1I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we obtain the equations

$$2x + y = 0$$

$$4x + 2y = 0$$

. Thus $y = -2x$, so a λ_2-eigenvector satisfies

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix}$$

. Choosing $x = 1$, we have found the (eigenvalue, eigenvector) pair $(1, \begin{bmatrix} 1 \\ -2 \end{bmatrix})$.

Now, our formula for $x(k)$ is $x(k) = c_1(\lambda_1)^k v_1 + c_2(\lambda_2)^k v_2 = c_1 5^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 1^k \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. To solve for c_1, c_2, we use our initial condition $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. By the formula, $x(0) = c_1(5)^0 v_1 + c_2 v_2 = c_1 v_1 + c_2 v_2$, so we obtain the system of equations $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, so that

$$1 = c_1 + c_2$$

$$0 = 2c_1 - 2c_2$$

. Adding 2 times equation 1 to equation 2, we find that $2 = 4c_1$ so $c_1 = \frac{1}{2}$. Since $0 = 2c_1 - 2c_2$, $c_1 = c_2 = \frac{1}{2}$. Thus the formula for $x(k) = A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is

$$x(k) = \frac{1}{2} 5^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

.

To find the formula for $A^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have the same values for λ_1, λ_2, v_1, v_2 (they depend only on A) but different constants c_1, c_2. Now, we solve the system $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$0 = c_1 + c_2$$

$$1 = 2c_1 - 2c_2$$

. Thus $c_1 = -c_2$, and so by substituting $-c_1$ for c_2, $1 = 2c_1 - 2c_2 = 2c_1 - 2(-c_1) = 4c_1$ implies $c_1 = \frac{1}{4}$, so $c_2 = -c_1 = -\frac{1}{4}$. Thus the formula for $x(k) = A^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is

$$x(k) = \frac{1}{4} 5^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

.

• Does the (Markov) matrix $M = \begin{bmatrix} 0 & 0.4 \\ 1 & 0.6 \end{bmatrix}$ satisfy the condition of for some n, all entries of M^n are positive? If so, what can we say about M?

Yes, note that

$$M^2 = \begin{bmatrix} 0 & 0.4 \\ 1 & 0.6 \end{bmatrix} \begin{bmatrix} 0 & 0.4 \\ 1 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.24 \\ 0.6 & 0.36 \end{bmatrix}$$

. All the entries of M^2 are positive, so it satisfies the condition. Thus, by a theorem, 1. there is a unique normalized equilibrium vector v, 2. M has the eigenvalue 1 and an eigenvalue $|\lambda| \leq 1$, and 3. for any vector $x(0)$ we have $x(k) = A^k x(0)$ converging to sv for $s = x_1(0) + x_2(0)$ (where v is the unique normalized equilibrium vector).